Metric Spaces and Topology
Lecture 24

Prop. Continuous functions map compact to compact spaces, i.e. for any continuous $f: X \to Y$, if X is compact, then so is $f(X)$ (in the relative top. of Y).

Proof. $\boxed{\text{HW}}$

Recall that in general, a continuous injection doesn't have to be an embedding, e.g. $id: (\mathbb{R}, \text{discrete}) \to (\mathbb{R}, \text{Euclidean})$.
However:

Cor. Any continuous injection from compact to Hausdorff is an embedding. In particular, any cont. inj. $2^{\mathbb{N}}$ into a Hausdorff space is an embedding.

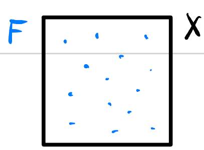
Proof. Let $f: X \to Y$ be cont. 1-to-1, X compact, Y Hausdorff.
We want to show that $f^{-1}: f(X) \to X$ is also continuous.
For this, we need to show that f maps open sets to relatively open subsets of $f(X)$. Because f is injective, f-images commute with complements, so it's enough

to show that f maps closed sets to closed subsets of Y. Let $C \subseteq X$ be closed, hence compact. Thus, $f(C)$ is also compact, and is thus closed because Y is Hausdorff. $\square$

<u>Sequential compactness.</u> A top space X is called *sequentially compact* if every sequence $(x_n) \subseteq X$ has a convergent subsequence. This is orthogonal to compactness in general top. spaces (i.e. neither implies, nor is implied by), however, sequential compactness coincides with compactness for metrizable spaces.

<u>Compactness for metric spaces.</u> For metric spaces, there is a third compact-like property we can formulate, and this too is equiv. to compactness:

<u>Heine-Borel property:</u> A metric space (X,d) is *Heine-Borel* if it is complete and totally bounded, where *totally bdd* means that $\forall \varepsilon > 0$, $\exists$ finite *ε-net*, i.e. a set $F \subseteq X$ s.t. $X \subseteq \bigcup_{x \in F} B_\varepsilon(x)$.



Theorem. For a metric space (X,d), TFAE:

(1) X is compact.

(2) X is sequentially compact.

(3) X is Heine–Borel, i.e. complete and totally bdd.

Remark. This implies that if X is compact metrizable, then any compatible metric on it is automatically complete and totally bdd (in particular bdd).

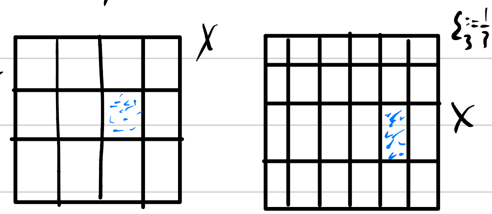
Proof. $(2) \Rightarrow (3)$. For completeness, take a Cauchy sequence. It has a convergent subseq., hence the whole Cauchy sequence must converge.

For total boundedness, let $\varepsilon > 0$ and suppose that there is no finite ε-net. Take a pt. $x_0 \in X$, thus $B_\varepsilon(x_0) \not\supseteq X$. Hence $\exists\, x_1 \in X \setminus B_\varepsilon(x_0)$ and $B_\varepsilon(x_0) \cup B_\varepsilon(x_1) \not\supseteq X$. Repeating this, we get $x_n \notin X \setminus \bigcup_{i<n} B_\varepsilon(x_i)$.

Thus we get a sequence $(x_n)_{n \in \mathbb{N}}$ whose elements are $\geq \varepsilon$ distance apart. This sequence has no Cauchy subsequence, hence no convergent subsequence, a contradiction. $\boxed{(2) \Rightarrow (3)}$

(3) $\Rightarrow$ (2). Let $(x_n) \subseteq X$ be a sequence. Because d is complete, it's enough to find a Cauchy subsequence.
Let $\varepsilon_n := \frac{1}{n}$ and for each $n \in \mathbb{N}$ $\exists$ finite ε_n-net F_n.
By the Pigeonhole Principle (this is König's lemma), we get a matrix of subsequences:

$$\begin{matrix} \boxed{x_{n_{11}}} & x_{n_{12}} & x_{n_{13}} & \cdots \\ x_{n_{21}} & \boxed{x_{n_{22}}} & x_{n_{23}} & \cdots \\ x_{n_{31}} & x_{n_{32}} & \boxed{x_{n_{33}}} & \cdots \\ \vdots & & & \end{matrix}$$



where each row is a subsequence of the previous row and the ith row is contained in an ε_i-ball. Then for any $i \in \mathbb{N}$, the diagonal sequence $(x_{n_{jj}})_{j \in \mathbb{N}}$ is eventually (starting from $j = i$) in an ε_i-ball, hence $(x_{n_{jj}})_{j \in \mathbb{N}}$ is Cauchy.

$\boxed{(3) \Rightarrow (2)}$

(1) $\Rightarrow$ (2). We prove the contrapositive: suppose there is a sequence $(x_n) \subseteq X$ with no convergent subsequence. Then no point $x \in X$ is a limit of a subsequence, hence $\forall$

$x \in X$, $\exists$ open neighbourhood $U_x \ni x$ s.t. $\exists$ at most finitely $x_n \in U_x$. These U_x form an open cover of X, so $\exists$ finite subcover $U_{y_1}, U_{y_2}, \dots, U_{y_k}$. But each U_{y_i} contains x_n for only finitely many $n \in \mathbb{N}$, contradicting that $\mathbb{N}$ is infinite. $\boxed{(1) \Rightarrow (2)}$

[(2) & (3)] => (1). Try doing this for $X = [0,1]$ first.